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# FINITE DIFFERENCE/GALERKIN SOLUTION FOR A MELTED MATERIAL LAYER TEMPERATURE FIELD

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## ABSTRACT

*This paper is a study of the temperature and velocity fields in a melted material layer. The Galerkin method as well as the finite difference method are used together to reduce the calculation time and the complexity of the treated problem. The paper is applying the finite difference/Galerkin method for the Benard-Marangoni convection case.*

## 1. Introduction

The temperature field in a melted material layer (a fluid layer) is a subject of study intensively treated by researchers studying not only the theoretical aspects of heat transfer processes but also the industrial applications of them [1÷9]. The presence of convection in a melted material layer will affect its homogeneity and, consequently, the physical and mechanical properties of the workpiece as well as its performance.

This paper is analyzing the case of a fluid layer with a small height, a situation where the surface tension is the main mechanism that is triggering the convection. This case is known in the literature as the Bénard-Marangoni convection.

Previous papers treated Bénard-Marangoni convection using the linear stability analysis method[1,2], the bifurcation analysis [10,11], the energy stability analysis, e.a. The advantages of each method related to the previous ones were related to the calculus simplicity and the analytical applicability.

This new method [12, 13, 14] is very simple and with less calculation effort compared to the previous ones. It uses a finite difference decomposition for vertical direction and Galerkin method (decomposition) for horizontal direction. Its application to the problem of Bénard-Marangoni convection is underlying once again the simplicity and efficiency of this method. It also give a new possibility of study of Bénard-Marangoni convection in a fluid layer and, further, for the active control of this convection type.

## 2. Governing equations

The mass, momentum and energy conservation equations for the fluid layer are (Chandrasekhar, 1981; Bejan, 1984):

$$\nabla \cdot v' = 0 \qquad (1)$$

$$\frac{\partial v'}{\partial t'} + v' \cdot \nabla v' = -\frac{1}{\rho_0}\nabla p' - \frac{\rho}{\rho_0} g\bar{z} + \frac{\mu}{\rho_0}\nabla^2 v' \qquad (2)$$

$$\rho c_p\left(\frac{\partial T'}{\partial t'} + v' \cdot \nabla T'\right) = k\nabla^2 T' \qquad (3)$$

where v is the velocity field in the horizontal direction (y), w is the velocity field in the vertical direction (z), t' is time , k is the fluid thermal conductivity, $\mu$ is the dynamic viscosity, $c_p$ is the specific heat, $\rho$ is the fluid density at temperature $T_0'$, $\rho_0$ is the fluid density at reference temperature $T_0'$, p is pressure, g is the gravitational acceleration.

I am using the following non-dimensional variables:

$$v = \frac{L}{\alpha}v', w = \frac{L}{\alpha}w', T = \frac{T'}{\frac{qL}{k}},\ p = \frac{L^2}{\alpha\nu\rho_0}p',$$
$$y = \frac{y'}{L},\ z = \frac{z'}{L},\ t = \frac{\alpha}{L^2}t' \qquad (4)$$

for velocity, temperature, pressure, length and time. Here, L is the fluid layer thickness $\alpha$ is the fluid layer thermal diffusivity, $\nu$ is the kinematics viscosity, q is the heat flux applied at the lower boundary.
The Boussinesq approximation imposes:

$$\rho = \rho_0 \left(1 - \beta(T - T_0)\right) \qquad (5)$$

where $\beta$ is the fluid volumetric expansion coefficient. The equations (4), (5) and the system of equations (1) ÷ (3), lead to the non-dimensional conservation equations:

$$\nabla \cdot v = 0 \qquad (6)$$

$$\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p^* - RaT\overline{z} + \nabla^2 v \qquad (7)$$

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = k\nabla^2 T \qquad (8)$$

where $p^* = p - p_{hydrostatic}$, Rayleigh number $Ra = \frac{\beta g q L^4}{k \alpha \nu}$, Prandtl number $Pr = \frac{\nu}{k}$.
Equation (7) can be written as a function of vorticity $\zeta = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$ in the following manner:

$$\frac{1}{Pr}\left(\frac{\partial \zeta}{\partial t} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z}\right) = Ra\frac{\partial T}{\partial y} + \nabla^2 \zeta \qquad (9)$$

In order to establish the equilibrium temperature field, I am considering Fourier cosine series decomposition for the temperature filed (T), the vertical (w) and horizontal (v) velocity fields [13,14]:

$$T(y,z) = T_0(z) + \sqrt{2}\sum_{m=1}^{N} T_m(z)\cos(a_k y) \qquad (10)$$

$$w(y,z) = \sqrt{2}\sum_{m=1}^{N} w_m(z)\cos(a_k y) \qquad (11)$$

$$v(y,z) = \sqrt{2}\sum_{m=1}^{N} v_m(z)\cos(a_k y) \qquad (12)$$

where , $a_k = \frac{2\pi k}{\lambda}, k = 1 \ldots M$, $D = \partial / \partial z$ and $\lambda$ is the ratio fluid length/height.
The orthogonal basis of the Galerkin procedure is: $1 \quad \sqrt{2}\sin(\alpha_m y) \quad \sqrt{2}\cos(\alpha_m y)$.
Next, the equations (10÷12) are substituted in the non-dimensional form of conservation equations (6-8). Averaging equation (6) over y direction, we obtain:

$$v_m = -\frac{Dw_m}{a_m}. \qquad (12)$$

Averaging equation (8) over y we obtain:

$$D^2 T_0 = \sum_{m=1}^{K}\left(Dw_m T_m + w_m DT_m\right) \qquad (13)$$

Multiplying equation (8) with $\sqrt{2}\cos(\alpha_m y)$ and averaging over y the following results were obtained:

$$D^2 T_m - (\xi w_{2m})DT_m - \left(\alpha_m^2 - \frac{\xi}{2}Dw_{2m}\right)T_m = DT_0 + \xi\left(2T_{2m}Dw_m + DT_{2m}w_m\right) + \xi f_1 \qquad (14)$$

with

$$f_1 = \sum_{k=1\neq m}^{K}\sum_{l=1\neq m}^{K}\left(\frac{\alpha_k TkDw_l}{\alpha_l} I_{klm}^{ssc}\right) + \sum_{k=1\neq m}^{K}\sum_{l=1\neq m}^{K}\left(DT_k w_l I_{klm}^{ccc}\right) \qquad (15)$$

Similarly, multiplying equation (9) with $\sqrt{2}\sin(\alpha_m y)$ and averaging over y, the modal velocity equation are:

$$D^4 w_m + (\phi w_{2m})D^3 w_{2m} + \left(\frac{\phi}{2}Dw_{2m} - 2\alpha_m^2\right) D^2 w_m + \left(\alpha_{2m}^2 \phi w_{2m} - \phi D^2 w_{2m} - \alpha_m^2 \phi w_{2m}\right) Dw_m + \left(\alpha_m^4 + \alpha_m \alpha_{2m}\phi Dw_{2m}\right)w_m + \left(-\frac{\phi}{2}D^3 w_{2m} - \frac{\alpha_m^2}{2}\phi Dw_{2m}\right)w_m = \alpha_m^2 RaT_m + \phi f_2 \qquad (16)$$

with

$$f_2 = \sum_{k=1\neq m}^{K}\sum_{l=1\neq m}^{K}\left(\frac{\alpha_k^2 w_k Dw_l}{\alpha_l} - \frac{D^2 w_k w_l}{\alpha_l}\right)I_{klm}^{css} + \sum_{k=1\neq m}^{K}\sum_{l=1\neq m}^{K} + \left(\frac{D^3 w_k w_l}{\alpha_l} - \alpha_k Dw_k w_l\right)I_{klm}^{scs} \qquad (17)$$

where $\phi = \frac{1}{\sqrt{2}\,Pr}$.

For the finite difference procedure, I used centered finite differences as indicated by McDonough and Catton [14]:

$$\left.\frac{dT}{dz}\right|_i = \frac{T^{i+1} - T^{i-1}}{2\Delta z} \quad (18)$$

$$\left.\frac{d^2T}{dz^2}\right|_i = \frac{T^{i+1} - 2T^i + T^{i-1}}{\Delta z^2} \quad (19)$$

$$\left.\frac{d^3T}{dz^4}\right|_i = \frac{T^{i+2} - 2T^{i+1} - 2T^{i-1} + T^{i-2}}{2\Delta z^3} \quad (20)$$

$$\left.\frac{d^4w}{dz^4}\right|_i = \frac{w^{i+2} - 4w^{i+1} + 6w^i - 4w^{i-1} + w^{i-2}}{\Delta z^4} \quad (21)$$

The equations (13), (14) and (16) were meshed using equations (18÷21). Consequently, we have to solve three systems of equations. Each system of equation has M unknowns, the number of points that I consider for the z (vertical) decomposition; the matrices are band (3 or 5 elements wide) matrices and a Gaussian elimination routine will be used.
In order to solve the systems of equations, the following boundary conditions were used:
— the no slip and no penetration upper and lower boundary conditions:

$$w_m\Big|_{z=0} = w_m\Big|_{z=1} = Dw_m\Big|_{z=0} = Dw_m\Big|_{z=1} \quad (22)$$

— the upper boundary heat transfer condition:

$$DT_m + BiT_m = 0\Big|_{z=1} \quad (23)$$

— the condition for surface tension driven convection:

$$D^2 w_m = Ma \cdot a^2 \cdot (T_m)\Big|_{z=1} \quad (24)$$

where $a = \frac{2\pi}{\lambda}$ is the wave number $a = 2\pi / \lambda$.

## 3. Numerical results

The numerical procedure was described by McDonoughtand Catton [14]. The initial guess was: $w_m$=0.0; $T_0 = 1 - z + sin(2\pi/\lambda)/100$, $T_m = sin(a_m z)/a_m$; After each iteration, the following solution convergence criteria is verified:

$$\begin{cases} \left|T_0^{new} - T_0^{old}\right|_{max} < \varepsilon \\ \left|T_m^{new} - T_m^{old}\right|_{max} < \varepsilon \\ \left|w_m^{new} - w_m^{old}\right|_{max} < \varepsilon \end{cases} \quad (25)$$

where $\varepsilon = 10^{-5}$. From one iteration to the other a relaxation procedure was used:

$$T^{new} = \delta T_m^{new} + (1-\delta) T_m^{old} \quad (26)$$

Throughout the paper δ=0.4, N=4, M=100, Biot number Bi=1.0, Marangoni number Ma=200.0, Prandtl number Pr=6.7; After 3 iterations the temperature modal values, $T_m$, are presented in figures 1÷4.

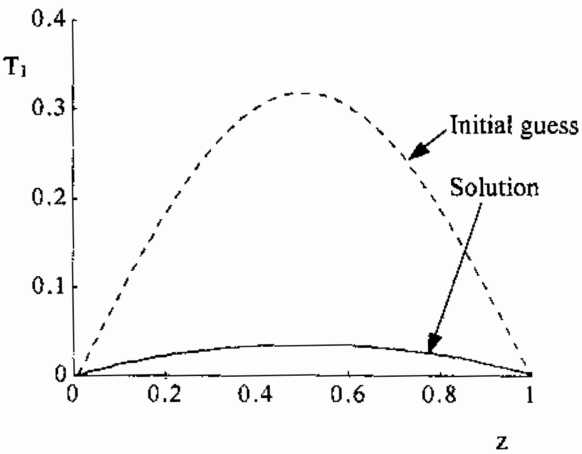


Fig. 1 $T_1$ – z variation

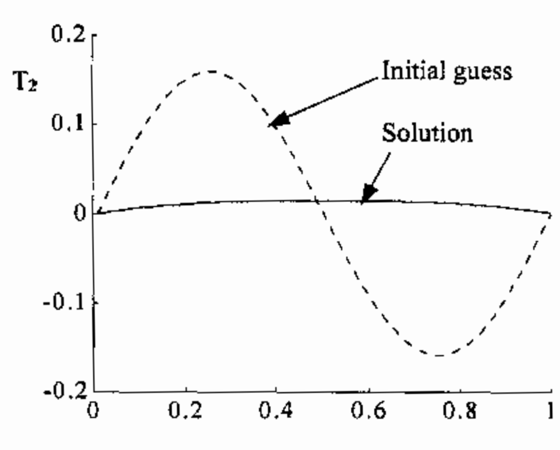


Fig.2 $T_2$ – z variation

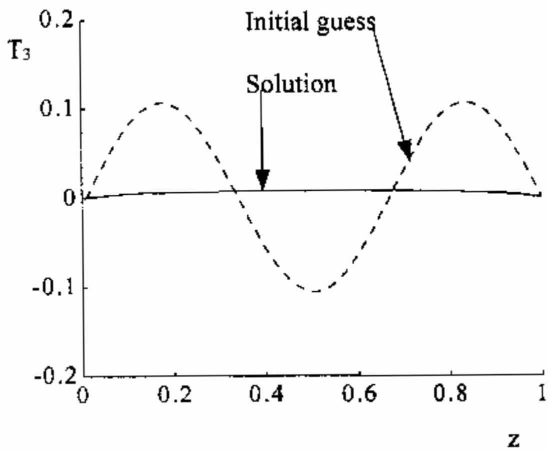


Fig.3 $T_3$ – z variation

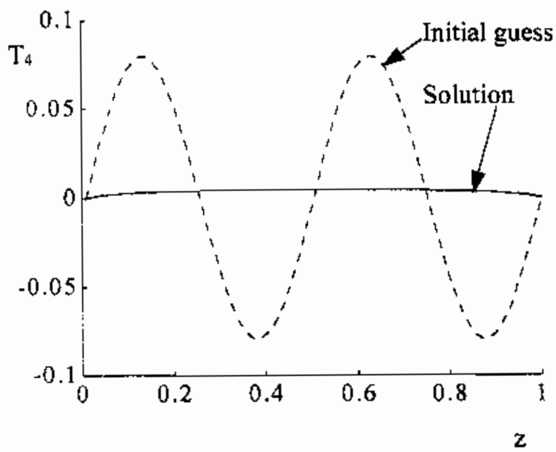


Fig.4 $T_4$ – z variation

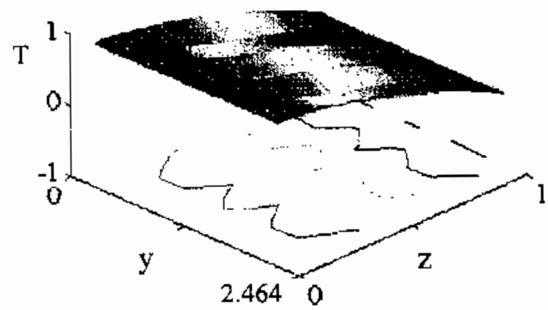


Fig.5 Temperature field for Bi=1.0, Ma=200.0, Pr=6.7, λ=2.464

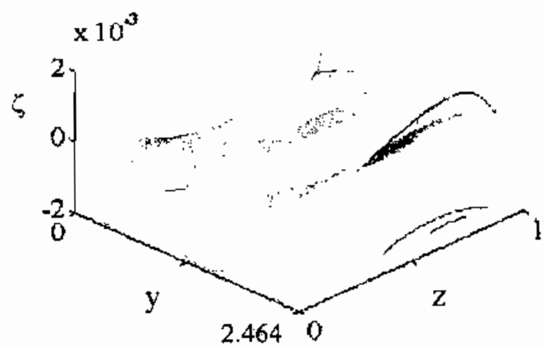


Fig.6 Vorticity field for Bi=1.0, Ma=200.0, Pr=6.7, λ=2.464.

Figurea 5 and 6 are presenting the temperature and the vorticity fields for the parameters presented above: Bi=1.0, Ma=200.0, Pr=6.7, λ=2.464.

The method is calculating iteratively the modal components of temperature and velocity fileds, $T_m$, $w_m$, $v_m$, m=1÷4, and in the end is establishing the values of the temperature and velocity fields using Fourier transformation, equations (10÷12).

Related to previous results [15,16], an very important advantage of this method is that we do not need to store the physical components of the temperature and velocity fields during calculation process and, consequently, we can have more iteration points and a higher precision for the results. An other advantage is the simplicity of the software needed for solving this problem, the decrease both of the calculation operations needed to be done and of the computation time.

The advantages presented above are indicating that this method could be used with success for the active control of Bénard-Marangoni convection. This method not only is giving us the whole picture of the convection process but also it is a very fast one having in view that finding and then tracking the solutions can be done very easy.

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# METODA "DIFERENTE FINITE/GALERKIN" PENTRU CAMPUL TERMIC AL UNUI STRAT DE MATERIAL TOPIT


**Rezumat**

*Această lucrare este un studiu al câmpului termic şi de viteze într-un strat de material topit. Metoda Galerkin şi metoda diferenţelor finite sunt folosite împreună pentru reducerea timpului de calcul şi a complexităţii problemei studiate. Lucrarea aplică metoda "diferenţe finite/Galerkin" pentru studiul convecţiei Bénard-Marangoni.*


# SOLUTION DE "DIFFERENCE FINIE/GALERKIN" POUR LA TEMPÉRATURE D'UNE COUCHE DE MATÉRIELLE FONDUE


Résumé

*Cet article est une étude des zones de la température et de vitesse dans une couche matérielle fondue. La méthode de Galerkin aussi bien que la méthode finie de différence sont employées ensemble pour réduire la période de calcul et la complexité du problème traité. Le papier applique la méthode finie de difference/Galerkin pour le cas de convection de Benard-Marangoni.*